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When are definable classes tilting and cotilting classes? ☆

Silvana Bazzoni

Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Trieste 63, 35121 Padova, Italy

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ABSTRACT

It is known that tilting classes are of finite type, while cotilting classes are not always of cofinite type. We investigate this phenomenon. By using a bijection between definable classes of left modules and definable classes of right modules, we prove that it reflects the asymmetry existing between the notions of covers and envelopes or, otherwise stated, right and left approximations.
In particular we show that there exist definable torsion classes containing the injective modules which are not tilting classes.

1. Introduction

The notion of definable classes arises in model theory. It is strictly related to the concept of purity which has a strong impact on the study of modules of infinite length. Definable classes have been formally introduced by Crawley-Boevey in [15]: they are the intersection of kernels of families of coherent functors. They are axiomatizable by first order formulas and are determined by the subclass of the pure-injective modules that they contain.

The interplay between definable classes, functor categories and purity has been worked out and developed, besides Crawley-Boevey, by Herzog [23], Krause [25] and Ziegler [35].

We are interested in particular instances of definable classes: the tilting and cotilting classes. The notion of tilting and cotilting modules, first introduced in the case of finite length modules over finite dimensional algebra, has gained importance in the more general setting of infinitely generated modules over arbitrary rings and it plays an important role in many problems formulated for the whole category of modules.

Recent papers [10] and [12] by Herbera, Šťovíček and the author have shown that tilting classes are of finite type, that is, they are the right Ext-orthogonal of families of compact modules. Since for every compact module S the functors $\mathrm{Ext}^i(S,-)$ are coherent, for every $i \in \mathbb{N}$, we conclude that

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E-mail address: bazzoni@math.unipd.it.

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tilting classes are definable. But they have the stronger property of being intersection of kernels of a particular kind of coherent functors, namely the Ext functors.

Thus a problem which arises is to understand which definable classes are tilting classes.

The cotilting case presents some similarities with the tilting case, but not every result valid for tilting classes has a counterpart for cotilting classes. For instance, as for the case of tilting classes, cotilting classes are definable. In fact, Šťovíček and the author [8,31] have shown that cotilting modules are pure injective, so a cotilting class is definable, since it is the left Ext-orthogonal of a cotilting module.

An asymmetry appears for the notion of finite type. In fact, while tilting classes are of finite type, cotilting classes are not of cofinite type in general, that is they are not the Tor-orthogonal of families of compact modules (cf. [9, Proposition 4.5]).

Our aim is to investigate this phenomenon. We will show that this asymmetry reflects the asymmetry existing between envelopes and covers. (See definitions in Section 2.)

More precisely, the result proved by Enochs and Xu [34, Theorem 2.2.12] stating that a precovering class closed under direct limits is a covering class, does not have a counterpart for preenveloping classes. For instance, there are examples of tilting classes, hence definable and special preenveloping, which are not enveloping (cf. [32, Theorem 3.5]).

We note that a definable class is a precovering class, hence a covering class by Enochs and Xu result. If it is moreover closed under extensions and contains the projective modules, then by Wakamatsu's Lemma, it is a special covering class.

On the opposite side, a result proved by Rada and Saorín [27, Corollary 3.5(c)] implies that a definable class is a preenveloping class, but in general it is not an enveloping class. Thus, we cannot apply Wakamatsu's Lemma to a definable class closed under extensions and containing the injective modules to conclude that it is a special preenveloping class.

In Theorem 6.3 we show that there exist definable classes $\mathcal{C}$ closed under extensions and even coresolving which are not special preenveloping. This will be achieved by showing that they are not tilting classes; in fact, tilting classes are special preenveloping classes. Moreover, if we impose on these classes $\mathcal{C}$ the condition to be special preenveloping, then we obtain that they are of finite type.

The main tool used will be the construction of a bijective map between definable classes of left modules and definable classes of right modules by which resolving classes correspond to coresolving classes and classes of cofinite type correspond to classes of finite type.

Thus the existence of cotilting classes which are not of cofinite type will imply the existence of classes $\mathcal{C}$ with the properties mentioned above, which are not tilting.

In particular we will show that there exist definable torsion classes containing the injective modules which are not tilting torsion classes.

The paper is organized as follows. In Sections 2 and 3 we introduce the definitions, the notations and some known results about approximation theory and tilting and cotilting theory. In Section 4 we recall the basic facts on functor categories and in particular the duality existing between the full subcategories of finitely presented functors on a ring and on its opposite.

Section 5 is a crucial part of the paper: we establish a correspondence between definable classes of left and right modules and we prove its properties.

In Section 6, using the correspondence defined in 5, we show that to every cotilting class not of cofinite type, a definable coresolving class closed under what we call n-images is associated which is not a tilting class, since it is not a special preenveloping class. Moreover, in Section 7 we specialize to the case of torsion and torsion free classes and give explicit constructions in the case of valuation domains.

We conclude in Section 8 with some questions concerning the existence of classes of rings admitting cotilting classes not of cofinite type.

2. Preliminaries

All rings considered are associative and with unit.

In this section we recall some definitions and we state the results that will be used later on.

We denote by Mod-R the category of all right R-modules and by mod-R, the subcategory of finitely presented modules. R-Mod and R-mod will denote the corresponding categories of left R-modules.

Definition 2.1. A class $\mathcal{C}$ of R-modules is *resolving* if it is closed under extensions, kernels of epimorphisms and contains the projective modules. Dually, $\mathcal{C}$ is *coresolving* if it is closed under extensions, cokernels of monomorphisms and contains the injective modules.

For every class $\mathcal{C}$ of R-modules, we let

$$\mathcal{C}^{\perp} = \{X \mid \operatorname{Ext}^i_R(C, X) = 0, \text{ for all } C \in \mathcal{C}, \text{ for all } i \geqslant 1\},$$
$$^{\perp}\mathcal{C} = \{X \mid \operatorname{Ext}^i_R(X, C) = 0, \text{ for all } C \in \mathcal{C}, \text{ for all } i \geqslant 1\},$$

and

$$\mathcal{C}^{\perp_1} = \{X \mid \operatorname{Ext}^1_R(C, X) = 0, \text{ for all } C \in \mathcal{C}\},$$
$$^{\perp_1}\mathcal{C} = \{X \mid \operatorname{Ext}^1_R(X, C) = 0, \text{ for all } C \in \mathcal{C}\}.$$

For a class $\mathcal{C}$ of right R-modules we let

$$\mathcal{C}^{\intercal} = \{X \in \text{Mod-}R \mid \operatorname{Tor}^R_i(C, X) = 0, \text{ for all } C \in \mathcal{C}, \text{ for all } i \geqslant 1\},$$
$$\mathcal{C}^{\intercal_1} = \{X \in \text{Mod-}R \mid \operatorname{Tor}^R_1(C, X) = 0, \text{ for all } C \in \mathcal{C}\},$$

and, analogously, for a class $\mathcal{C}$ of left R-modules, we will use the notation $^{\intercal}\mathcal{C}$ and $^{\intercal_1}\mathcal{C}$.

A pair $(\mathcal{A}, \mathcal{B})$ of classes of modules, is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. If $\mathcal{A}$ is a class of right R-modules and $\mathcal{B}$ a class of left R-modules, the pair $(\mathcal{A}, \mathcal{B})$ is a Tor-*pair* provided that $\mathcal{A} = {}^{\intercal_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\intercal_1}$.

For every class $\mathcal{C}$, $^{\perp}\mathcal{C}$ is a resolving class and in particular, it is syzygy-closed. Dually, $\mathcal{C}^{\perp}$ is coresolving and cosyzygy-closed.

Note that if $\mathcal{C}$ is resolving, then $\mathcal{C}^{\perp} = \mathcal{C}^{\perp_1}$ and $\mathcal{C}^{\intercal} = \mathcal{C}^{\intercal_1}$; if $\mathcal{C}$ is coresolving, then $^{\perp}\mathcal{C} = {}^{\perp_1}\mathcal{C}$. A pair $(\mathcal{A}, \mathcal{B})$ is called a *hereditary cotorsion pair* if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. It is easy to see that $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair and $\mathcal{A}$ is resolving if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair and $\mathcal{B}$ is coresolving.

A concept very useful when dealing with cotorsion pairs is the notion of approximations via precovers and preenvelopes defined by Enochs in [19] as a generalization of the notion of right and left approximations introduced by Auslander and Smalø [6] for finite dimensional algebras.

We recall now the definitions.

Let $\mathcal{C}$ be a class of R-modules and let M be an R-module. A morphism $\phi: C \to M$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-*precover* of M if $\operatorname{Hom}(C', \phi): \operatorname{Hom}_R(C', C) \to \operatorname{Hom}_R(C', M)$ is surjective, for every $C' \in \mathcal{C}$. A $\mathcal{C}$-precover is a $\mathcal{C}$-*cover* if it is minimal in the sense that any endomorphism f of C such that $\phi \circ f = \phi$ is an isomorphism. A $\mathcal{C}$-precover $\phi: C \to M$ is called *special* if it is an epimorphism and $\operatorname{Ker}\phi \in \mathcal{C}^{\perp_1}$.

Dually a morphism $\phi: M \to C$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-*preenvelope* of M if $\operatorname{Hom}(\phi, C'): \operatorname{Hom}_R(M, C') \to \operatorname{Hom}_R(C, C')$ is surjective, for every $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope is a $\mathcal{C}$-*envelope* if any endomorphism f of C such that $f \circ \phi = \phi$ is an isomorphism. A $\mathcal{C}$-preenvelope $\phi: C \to M$ is called *special* if it is a monomorphism and $\operatorname{Coker}\phi \in {}^{\perp_1}\mathcal{C}$.

A class $\mathcal{C}$ is said to be a *precovering*, *covering*, *special precovering* class (*preenveloping*, *enveloping*, *special preeneveloping* class) if every R-module admits a $\mathcal{C}$-precover, $\mathcal{C}$-cover, special $\mathcal{C}$-precover ($\mathcal{C}$-preenvelope, $\mathcal{C}$-envelope, special $\mathcal{C}$-preenvelope).

We recall now some important results on approximation theory. They relate the existence of $\mathcal{C}$-approximations to closure properties of the class $\mathcal{C}$.

The next theorem is due to Enochs and Xu.

Theorem 2.2. *(See [34, Theorem 2.2.12].) If a class $\mathcal{C}$ of modules is closed under direct limits and it is a precovering class, then it is a covering class.*

Lemma 2.3 (Wakamatsu's Lemma). *Let $\mathcal{C}$ be a class of modules closed under extensions. If $\phi : C \to M$ is a surjective $\mathcal{C}$-cover of M, then ϕ is a special $\mathcal{C}$-cover.*

For the case of preenveloping classes, Rada and Saorín proved the following.

Theorem 2.4. *(See [27, Corollary 3.5(c)].) If a class $\mathcal{C}$ of modules is closed under direct products and pure submodules, then it is a preenveloping class.*

A slight elaboration of the arguments used by Bican, El Bashir and Enochs to prove the flat cover conjecture, furnish the following result which is a dual version of Theorem 2.4 (see also [24]).

Theorem 2.5. *(See [20, Proposition 5.2.2], [13, Theorems 5 and 6].) If a class of modules is closed under direct sums and pure epimorphic images, then it is a precovering class.*

An immediate consequence is

Corollary 2.6. *If a class of modules is closed under direct sums and pure epimorphic images, then it is a covering class.*

Proof. Since the class is closed under direct sums and pure epimorphic images, it is closed also under direct limit. Thus, Theorem 2.5 and Theorem 2.2 give the conclusion. □

3. *n*-tilting and *n*-cotilting classes

For every $n \in \mathbb{N}$ we denote by $\mathcal{P}_n$ the class of modules with projective dimension at most n and by $\mathcal{I}_n$ the class of modules with injective dimension at most n.

Definition 3.1. Let $n \in \mathbb{N}$. A module T is *n-tilting* provided

(T1) $T \in \mathcal{P}_n$,
(T2) $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geqslant 1$ and all sets I, and
(T3) there exists $r \geqslant 0$ and a long exact sequence

$$0 \to R \to T_0 \to \cdots \to T_r \to 0$$

such that $T_i \in \operatorname{Add} T$ for each $0 \leqslant i \leqslant r$.

Here, $\operatorname{Add} T$ denotes the class of all direct summands of arbitrary direct sums of copies of T.
The class $T^{\perp}$ is called n-tilting class.

There are various characterizations of n-tilting classes. We recall the following.

Theorem 3.2. *(See [1], [22, Theorem 5.1.14].) Let $\mathcal{C}$ be a class of R-modules and let $n \in \mathbb{N}$. The following are equivalent*:

(1) *$\mathcal{C}$ is n-tilting.*
(2) *$\mathcal{C}$ is coresolving, special preenveloping, closed under direct sums and direct summands and ${}^{\perp}\mathcal{C}$ is contained in $\mathcal{P}_n$.*

We have also dual definitions.

Definition 3.3. Let $n \in \mathbb{N}$. A module C is *n-cotilting* provided

(C1) $C \in \mathcal{I}_n$,
(C2) $\mathrm{Ext}^i_R(C^I, C) = 0$ for each $i \geqslant 1$ and all sets I, and
(C3) there exists $r \geqslant 0$ and a long exact sequence

$$0 \to C_r \to \cdots \to C_0 \to W \to 0$$

such that $C_i \in \mathrm{Prod}\, C$ for each $0 \leqslant i \leqslant r$ and W is an injective R-cogenerator.

Here, $\mathrm{Prod}\, C$ denotes the class of all direct summands of arbitrary direct products of copies of C. The class ${}^{\perp}C$ is called n-cotilting class.

A characterization of n-cotilting classes is given by the following.

Theorem 3.4. *(See [1], [22, Theorem 8.1.9].) Let $\mathcal{C}$ be a class of R-modules and let $n \in \mathbb{N}$. The following are equivalent*:

(1) *$\mathcal{C}$ is n-cotilting.*
(2) *$\mathcal{C}$ is resolving, covering, closed under direct products and direct summands and $\mathcal{C}^{\perp}$ is contained in $\mathcal{I}_n$.*

To illustrate other characterizations of n-tilting and of n-cotilting classes, we first need to give a restriction of the notion of finitely presented modules.

Definition 3.5. We say that an R-module S is *compact* if it admits a projective resolution consisting of finitely generated projective modules.

Note that, if the ring R is noetherian, then a module is compact if and only if it is finitely generated and over an arbitrary ring a module of projective dimension at most one is compact if and only if it is finitely presented.

We will consider the notion of finite type and cofinite type for classes of modules as it was introduced in [2].

Definition 3.6. (See [2].) Let $\mathcal{C}$ be a class of R-modules and let $n \in \mathbb{N}$.

$\mathcal{C}$ is said to be of *finite type* if there exists a set $\mathcal{S}$ of compact modules, such that $\mathcal{S} \subseteq \mathcal{P}_n$ and $\mathcal{C} = \mathcal{S}^{\perp}$. A class $\mathcal{C}$ of left modules is said to be of *cofinite type* if there exists a set $\mathcal{S}$ of compact right modules, such that $\mathcal{S} \subseteq \mathcal{P}_n$ and $\mathcal{C} = \mathcal{S}^{\intercal}$.

Remark 1. Note that a class $\mathcal{C}$ is of finite type if and only if $\mathcal{C} = \mathcal{S}^{\perp}$, where $\mathcal{S}$ is the set of compact modules in $\mathcal{P}_n \cap {}^{\perp}\mathcal{C}$; and a class $\mathcal{C}$ is of cofinite type if and only if $\mathcal{C} = \mathcal{S}^{\intercal}$ where $\mathcal{S}$ is the set of compact modules in $\mathcal{P}_n \cap {}^{\intercal}\mathcal{C}$.

We illustrate now the asymmetry existing between the concepts of finite and cofinite type.

Proposition 3.7. *Let R be a ring. The following hold*:

(1) *A class $\mathcal{C}$ of R-modules is of finite type if and only it is an n-tilting class.*
(2) *If a class $\mathcal{C}$ of R-modules is of cofinite type, then it is an n-cotilting class. There exist n-cotilting classes which are not of cofinite type.*

Proof. The fact that every n-tilting class is of finite type is a crucial result which was proved first for 1-tilting classes in [10] and for the general case in [12].

Conversely, if $\mathcal{C}$ is a class of finite type, then $\mathcal{C}$ is the right Ext-orthogonal of a set of compact modules contained in $\mathcal{P}_n$, thus $\mathcal{C}$ is coresolving, closed under direct sums and direct summands. Moreover, by [18, Theorem 10] $\mathcal{C}$ is a special preenveloping class and by [1, Lemma 2.2], ${}^{\perp}\mathcal{C} \subseteq \mathcal{P}_n$. Thus $\mathcal{C}$ is an n-tilting class by Theorem 3.2.

(2) Assume that $\mathcal{C}$ is of cofinite type and let $\mathcal{S} \subseteq \mathcal{P}_n$ be a set of compact modules such that $\mathcal{C} = \mathcal{S}^{\intercal}$. Then $\mathcal{C} = {}^{\perp}\mathcal{S}^*$, where $\mathcal{S}^*$ is the set of the character modules of the modules in $\mathcal{S}$. Then, $\mathcal{S}^* \subseteq \mathcal{I}_n$. Since every character module is pure injective, [17, Corollary 10] implies that $\mathcal{C}$ is a covering class. Moreover, $\mathcal{C}$ is resolving and by [1, Lemma 2.2] $\mathcal{C}^{\perp} \subseteq \mathcal{I}_n$. So $\mathcal{C}$ is an n-cotilting class, by Theorem 3.4.

In [9] it is shown that there exist 1-cotilting classes which are not of cofinite type. □

4. The functor category and definable classes

The functor category is the category

$$\mathcal{C}_R := (R\text{-mod}, Ab)$$

whose objects are the additive functors $F : R\text{-mod} \to Ab$ (Ab denotes the category of abelian groups) and whose morphisms are the natural transformations. It is an abelian category where the kernels and cokernels are defined pointwise.

In [23] it is illustrated the role of the category $\mathcal{C}_R$ for the study of the category Mod-R.

The Yoneda embedding $H : R\text{-mod} \to \mathcal{C}_R$ associates to every object $X \in R$-mod, the functor $H_X = \mathrm{Hom}_R(X, -)$ and to a morphism $\theta : X \to Y$ in R-mod the morphism $\mathrm{Hom}_R(\theta, -) : \mathrm{Hom}_R(Y, -) \to \mathrm{Hom}_R(X, -)$. The functor H_X will be denoted also by $(X, -)$ and called a representable functor; the morphism $\mathrm{Hom}_R(\theta, -)$ will be also denoted by H_θ.

Other examples of functors in $\mathcal{C}_R$ are the functors T_M associated to right R-modules M. For every object $X \in R$-mod, $T_M(X) = M \otimes_R X$ and for every morphism $\theta : X \to Y$ in R-mod, T_θ is the morphism $1_M \otimes_R \theta : M \otimes_R X \to M \otimes_R Y$.

The assignment $M \to T_M$ is a right exact fully faithful functor from the category Mod-R to the category $\mathcal{C}_R$.

Remark 2. The functors H_X and T_M defined above act also on arbitrary left R-modules. Thus, the associations ${}_RX \mapsto H_X$; $\theta \mapsto H_\theta$ and $M_R \mapsto T_M$; $\theta \mapsto T_\theta$ can be viewed as embeddings into the "large" category of all the functors from R-Mod to Ab. It is a large category in the sense that the morphisms between two objects in the category do not form a set.

A functor $F \in \mathcal{C}_R$ is *finitely generated* if there exists an exact sequence $(X, -) \to F \to 0$, for some $X \in R$-mod. The functors $(X, -)$ are the finitely generated projective objects of $\mathcal{C}_R$.

A functor $F \in \mathcal{C}_R$ is *finitely presented* if there is an exact sequence

$$(Y, -) \to (X, -) \to F \to 0,$$

for some $X, Y \in R$-mod. By Yoneda's Lemma, F is finitely presented if and only if there is a morphism $\theta : X \to Y$ in R-mod such that $F = \mathrm{Coker}\, H_\theta$.

Moreover, a functor $F \in \mathcal{C}_R$ is *coherent* if it is finitely presented and every finitely generated subobject is finitely presented.

Denote by fp-$\mathcal{C}_R$ the full subcategory of $\mathcal{C}_R$ consisting of the finitely presented objects. It has been proved by Auslander [3] that fp-$\mathcal{C}_R$ is an abelian category, namely that every finitely presented object is coherent. Thus the coherent objects of $\mathcal{C}_R$ are the functors $\mathrm{Coker}\, H_\theta$, for some $\theta \in R$-mod.

All the considerations made above for left R-modules can be transfered to the case of right R-modules; that is, we can define the functor category ${}_R\mathcal{C}$ consisting of the additive functors

$F : \text{mod-}R \to Ab$ and its full subcategory fp-${}_R\mathcal{C}$. All the results mentioned above are valid for the category ${}_R\mathcal{C}$ with the obvious modifications.

We recall now the duality existing between the left and right functor categories.

To every additive functor $F \in \mathcal{C}_R$ one can associate a functor

$$F^\vee : \text{Mod-}R \to Ab$$

defined by

$$F^\vee(M) = \operatorname{Hom}_{\mathcal{C}_R}(F, T_M),$$

for every right R-module M.

There is an analogous formula to define $F^\vee$ for every functor in $F \in {}_R\mathcal{C}$.

Auslander [4] and Gruson and Jensen [21] noted that the correspondence $^\vee$ defined above gives a duality between fp-$\mathcal{C}_R$ and fp-${}_R\mathcal{C}$ as it is stated below.

For a clear proof of this duality we refer to the paper by Herzog [23, §5]. See also [25, Ch. 2].

Theorem 4.1. *(See [4, §7], [21].) The functor*

$$F \mapsto F^\vee$$

induces a duality between the category fp-$\mathcal{C}_R$ *and the category* fp-${}_R\mathcal{C}$*. Moreover, the following hold*

(1) *For every $X \in R$-mod and every $A \in$ mod-R, we have*

$$(H_X)^\vee = T_X; \qquad (T_A)^\vee = H_A.$$

(2) *For every $\theta : X \to Y$ in R-mod and every $\phi \in$ mod-R, we have*

$$(\operatorname{Coker} H_\theta)^\vee = \operatorname{Ker} T_\theta; \qquad (\operatorname{Ker} T_\phi)^\vee = \operatorname{Coker} H_\phi.$$

For functors defined on the whole module category and for classes of R-modules, we recall the following definition.

Definition 4.2.

(1) A functor $F : R\text{-Mod} \to Ab$ (or $F : \text{Mod-}R \to Ab$) is called *c*oherent if it commutes with direct limits and direct products.
(2) A class of modules is called *definable* if it is closed under arbitrary direct products, direct limits, and pure submodules.

Crawley-Boevey characterized coherent functors and definable classes as follows.

Proposition 4.3. *(See [15, §2.1, §2.3].)*

(1) *A functor $F : R\text{-Mod} \to Ab$ is coherent if and only if it is isomorphic to* $\operatorname{Coker} H_\theta$ *for a morphism $\theta : X \to Y$ in R-mod.*
(2) *A class of modules $\mathcal{C}$ is definable if and only if it is the intersection of the kernels of a set of coherent functors.*
(3) *In particular, definable classes are closed under pure epimorphic images.*

Proof. Statements (1) and (2) are proved in [15].

(3) follows immediately by (1) and (2), since finitely presented modules have the projective property with respect to pure epimorphisms. $\square$

Another important approach to definable classes is via model theory of modules.

An R-module is a *model* of a theory in the first order language $\mathcal{M}(R)$ of R-modules and there is a relevant connection between coherent functors and formulas in the language $\mathcal{M}(R)$.

A *positive primitive formula* (ppf) in the language $\mathcal{M}(R)$ (of left R-modules) is a formula of the form $\phi(\mathbf{x}) \Leftrightarrow \exists \mathbf{y} A\mathbf{x} + B\mathbf{y} = 0$ where A and B are matrices with entries in R and $\mathbf{x}, \mathbf{y}$ are column tuples. Given a ppf formula $\phi(\mathbf{x})$ and $M \in R$-Mod, the assignment

$$\phi(M) = \{\mathbf{a} \in M^n \mid M \models \phi(\mathbf{a})\}$$

describes all the coherent subfunctors of the *n-fold forgetful functor*

$$\mathrm{Forget}^n : M \to M^n.$$

Thus, in Proposition 4.3, the following characterization of a definable class $\mathcal{C}$ of modules can be added:

(2′) $\mathcal{C}$ is defined in the first order language $\mathcal{M}(R)$ by satisfying $\phi(\mathbf{x}) \to \psi(\mathbf{y})$, for ppf formulas $\phi(\mathbf{x})$ and $\psi(\mathbf{y})$.

Recalling that two modules are elementary equivalent if they satisfy the same sentences (formulas without free variables) condition (2′) shows that definable classes are closed under elementary equivalence. In particular this implies the important fact that definable classes are determined by the subclass of the pure-injective modules that they contain, since every module is elementary equivalent to its pure-injective hull (see [29]).

For more results and details on the subject, see [15, §2], [26] and [35, §1].

5. A correspondence between definable classes of right and left modules

In this section we make use of the duality mentioned in Section 4, to define a correspondence between definable classes of left R-modules and definable classes of right R-modules.

Proposition 5.1. *Let R be a ring and let ${}_R\mathcal{F}$ be a class of left R-modules. The following are equivalent.*

(1) *${}_R\mathcal{F}$ is a definable class.*
(2) *There exists a set* $\{\theta_i : X_i \to Y_i \mid \theta_i \in R\text{-mod}\}$ *such that*

$${}_R\mathcal{F} = \{N \in R\text{-Mod} \mid \mathrm{Coker}\, H_{\theta_i}(N) = 0, \textit{ for all } \theta_i\}.$$

(3) *There exists a set* $\{\phi_i : A_i \to B_i \mid \phi_i \in \text{mod-}R\}$ *such that*

$${}_R\mathcal{F} = \{N \in R\text{-Mod} \mid \mathrm{Ker}\, T_{\phi_i}(N) = 0, \textit{ for all } \phi_i\}.$$

Proof. (1) $\Leftrightarrow$ (2) follows by Proposition 4.3.

(2) $\Leftrightarrow$ (3) Let $F_i = \mathrm{Coker}\, H_{\theta_i}$. Then $F_i^\vee$ belongs to fp-${}_R\mathcal{C}$, by Theorem 4.1. Thus there is $\phi_i : A_i \to B_i \in$ mod-R such that $F_i^\vee = \mathrm{Coker}\, H_{\phi_i}$. Again by duality, we conclude that $F_i = \mathrm{Ker}\, T_{\phi_i}$. The converse holds by dual arguments. $\square$

The analogous result holds for classes of right R-modules, that is:

Proposition 5.2. *Let R be a ring and let $\mathcal{T}_R$ be a class of right R-modules. The following are equivalent.*

(1) *$\mathcal{T}_R$ is a definable class.*
(2) *There exists a set $\{\phi_i : A_i \to B_i \mid \phi_i \in \text{mod-}R\}$ such that*

$$\mathcal{T}_R = \left\{M \in \text{Mod-}R \;\middle|\; \operatorname{Coker} H_{\phi_i}(M) = 0,\ \textit{for all } \phi_i\right\}.$$

(3) *There exists a set $\{\theta_i : X_i \to Y_i \mid \theta_i \in R\text{-mod}\}$ such that*

$$\mathcal{T}_R = \left\{M \in \text{Mod-}R \;\middle|\; \operatorname{Ker} T_{\theta_i}(M) = 0,\ \textit{for all } \theta_i\right\}.$$

We fix now a setting for the definition of the correspondence between definable classes of left and right R-modules.

Definition 5.3. Let ${}_R\mathcal{F}$ be a definable class of left R-modules. By Proposition 5.1, there is a set $\Phi = \{\phi_i : A_i \to B_i \mid \phi_i \in \text{mod-}R\}$ such that ${}_R\mathcal{F} = \{N \in R\text{-Mod} \mid \operatorname{Ker} T_{\phi_i}(N) = 0\}$. Without loss of generality we can assume that Φ is the set of all morphisms $\phi \in \text{mod-}R$ such that ${}_R\mathcal{F} = \{N \in R\text{-Mod} \mid \operatorname{Ker} T_{\phi}(N) = 0\}$.

Consider the correspondence

$${}_R\mathcal{F} \stackrel{\rho}{\longmapsto} \mathcal{T}_R$$

defined by

$$\mathcal{T}_R = \left\{M \in \text{Mod-}R \;\middle|\; \operatorname{Coker} H_{\phi}(M) = 0,\ \text{for all } \phi \in \Phi\right\}.$$

Analogously, let $\mathcal{T}_R$ be a definable class of right R-modules and let $\Phi = \{\phi_i : A_i \to B_i \mid \phi_i \in \text{mod-}R\}$ be a set of morphisms such that $\mathcal{T}_R = \{M \in \text{Mod-}R \mid \operatorname{Coker} H_{\phi_i}(M) = 0\}$. Without loss of generality we can assume that Φ is the set of all morphisms $\phi \in \text{mod-}R$ such that $\mathcal{T}_R = \{M \in \text{Mod-}R \mid \operatorname{Coker} H_{\phi}(M) = 0\}$.

Consider the correspondence

$$\mathcal{T}_R \stackrel{\sigma}{\longmapsto} {}_R\mathcal{F}$$

defined by

$${}_R\mathcal{F} = \left\{N \in R\text{-Mod} \;\middle|\; \operatorname{Ker} T_{\phi}(N) = 0,\ \text{for all } \phi \in \Phi\right\}.$$

Propositions 5.1 and 5.2 give:

Proposition 5.4. *The correspondences ρ and σ of Definition 5.3 are mutually inverse and they give a bijection between the collection of definable classes of left R-modules and the collection of definable classes of right R-modules.*

We show now a very useful property of the correspondences ρ and σ. It will be crucial in the sequel. For every R-module C we denote by C^* the character module, that is $\operatorname{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$.

Lemma 5.5. *Let R be a ring and consider a set $\mathcal{G}$ of morphisms in* mod-R,

$$\mathcal{G} = \{\phi_i : A_i \to B_i \mid \phi_i \in \text{mod-}R\}.$$

Let

$$_R\mathcal{F} = \big\{N \in R\text{-Mod} \bigm| \operatorname{Ker} T_{\phi_i}(N) = 0,\ \textit{for all } \phi_i \in \mathcal{G}\big\}$$

and

$$\mathcal{T}_R = \big\{M \in \text{Mod-}R \bigm| \operatorname{Coker} H_{\phi_i}(M) = 0,\ \textit{for all } \phi_i \in \mathcal{G}\big\}.$$

Then, the following hold:

(1) *$N \in {_R\mathcal{F}}$ if and only if $N^* \in \mathcal{T}_R$.*
(2) *$M \in \mathcal{T}_R$ if and only if $M^* \in {_R\mathcal{F}}$.*

Proof. (1) Let $N \in {_R\mathcal{F}}$. Then for every $\phi_i \in \mathcal{G}$ the sequence

$$0 \to A_i \otimes_R N \xrightarrow{\phi_i \otimes 1_N} B_i \otimes_R N$$

is exact. Dualizing we get the exact sequence

$$(B_i \otimes_R N)^* \xrightarrow{(\phi_i \otimes 1_N)^*} (A_i \otimes_R N)^* \to 0.$$

By the adjunction between the Hom and tensor functors we have that the sequence

$$\operatorname{Hom}_R\big(B_i, N^*\big) \xrightarrow{\operatorname{Hom}(\phi_i, N^*)} \operatorname{Hom}_R\big(A_i, N^*\big) \to 0$$

is exact. Hence, $\operatorname{Coker} H_{\phi_i}(N^*) = 0$, that is $N^* \in \mathcal{T}_R$.

Conversely, let N be a left R-module such that $N^* \in \mathcal{T}_R$. Then for every $\phi_i \in \mathcal{G}$ we have an exact sequence:

$$\operatorname{Hom}_R\big(B_i, N^*\big) \xrightarrow{\operatorname{Hom}(\phi_i, N^*)} \operatorname{Hom}_R\big(A_i, N^*\big) \to 0.$$

By adjunction, the sequence

$$(B_i \otimes_R N)^* \xrightarrow{(\phi_i \otimes 1_N)^*} (A_i \otimes_R N)^* \to 0$$

is exact. Passing to the duals we get the exact sequence

$$0 \to (A_i \otimes_R N)^{**} \xrightarrow{(\phi_i \otimes 1_N)^{**}} (B_i \otimes_R N)^{**}.$$

Since every module is canonically embedded in its double dual, we conclude that $0 \to A_i \otimes_R N \xrightarrow{\phi_i \otimes 1_N} B_i \otimes_R N$ is exact. Hence, $\operatorname{Ker} T_{\phi_i}(N) = 0$, for every $\phi_i \in \mathcal{G}$ and so $N \in {_R\mathcal{F}}$.

(2) Let $M \in \mathcal{T}_R$. Then, for every $\phi_i \in \mathcal{G}$ the sequence

$$\operatorname{Hom}_R(B_i, M) \xrightarrow{\operatorname{Hom}(\phi_i, M)} \operatorname{Hom}_R(A_i, M) \to 0$$

is exact. Dualizing we get the exact sequence

$$0 \to \big(\operatorname{Hom}_R(A_i, M)\big)^* \xrightarrow{(\operatorname{Hom}(\phi_i, M))^*} \big(\operatorname{Hom}_R(A_i, M)\big)^*.$$

From the canonical isomorphism $C \otimes_R M^* \cong (\mathrm{Hom}_R(C, M))^*$ valid for every finitely presented right R-module C and every right R-module M, we have that the sequence

$$0 \to A_i \otimes_R M^* \xrightarrow{\phi_i \otimes 1_{M^*}} B_i \otimes_R M^*$$

is exact, thus $\mathrm{Ker}\, T_{\phi_i}(M^*) = 0$, that is $M^* \in {}_R\mathcal{F}$.

Conversely, let M be a right R-module such that $M^* \in {}_R\mathcal{F}$. Then for every $\phi_i \in \mathcal{G}$ the sequence

$$0 \to A_i \otimes_R M^* \xrightarrow{\phi_i \otimes 1_{M^*}} B_i \otimes_R M^*$$

is exact. Thus, from the canonical isomorphism the sequence

$$0 \to \big(\mathrm{Hom}_R(A_i, M)\big)^* \xrightarrow{(\mathrm{Hom}(\phi_i, M))^*} \big(\mathrm{Hom}_R(A_i, M)\big)^*,$$

is exact, that is $(\mathrm{Hom}(\phi_i, M))^*$ is a monomorphism.

We want to show that $\mathrm{Hom}_R(B_i, M) \xrightarrow{\mathrm{Hom}(\phi_i, M)} \mathrm{Hom}_R(A_i, M)$ is an epimorphism for every $\phi_i \in \mathcal{G}$. Consider the exact sequence

$$\mathrm{Hom}_R(B_i, M) \xrightarrow{\mathrm{Hom}(\phi_i, M)} \mathrm{Hom}_R(A_i, M) \to \mathrm{Coker}\, H_{\phi_i} \to 0.$$

Passing to its dual sequence we get the exact sequence

$$0 \to (\mathrm{Coker}\, H_{\phi_i})^* \to \big(\mathrm{Hom}_R(A_i, M)\big)^* \xrightarrow{(\mathrm{Hom}(\phi_i, M))^*} \big(\mathrm{Hom}_R(A_i, M)\big)^*.$$

But, $(\mathrm{Hom}(\phi_i, M))^*$ is a monomorphism, so $(\mathrm{Coker}\, H_{\phi_i})^* = 0$, hence also $\mathrm{Coker}\, H_{\phi_i} = 0$ showing that $M \in \mathcal{T}_R$. $\square$

For convenience we introduce a generalization of the notion of submodules and of epimorphic images.

Definition 5.6. Let $n \geqslant 1$ and let $\mathcal{C}$ be a class of R-modules. We say that an R-module A is an *n-submodule* in $\mathcal{C}$ if there is an exact sequence

$$0 \to A \to C_{n-1} \to \cdots \to C_1 \to C_0$$

with $C_i \in \mathcal{C}$, for every $0 \leqslant i \leqslant n-1$.

Dually, B is an *n-image* in $\mathcal{C}$ if there is an exact sequence

$$C_{n-1} \to \cdots \to C_1 \to C_0 \to B \to 0$$

with $C_i \in \mathcal{C}$, for every $0 \leqslant i \leqslant n-1$.

Note that 1-submodules (1-images) in $\mathcal{C}$ are just the submodules (epimorphic images) of modules in $\mathcal{C}$.

Proposition 5.7. *Let R be a ring. Let ${}_R\mathcal{F}$ be a definable class of left R-modules and let $\mathcal{T}_R$ be a definable class of right R-modules. Consider the correspondences ρ and σ of Definition* 5.3. *Then, the following hold*:

(1) *${}_R\mathcal{F}$ is resolving if and only if $\rho({}_R\mathcal{F})$ is coresolving.*

(2) *$\mathcal{T}_R$ is coresolving if and only if $\sigma(\mathcal{T}_R)$ is resolving.*
(3) *${}_R\mathcal{F}$ is closed under n-submodules if and only if $\rho({}_R\mathcal{F})$ is closed under n-images.*
(4) *$\mathcal{T}_R$ is closed under n-images if and only if $\sigma(\mathcal{T}_R)$ is closed under n-submodules.*

Proof. It is enough to prove (1) and (3), since (2) and (4) will follow by them and by Proposition 5.4.

(1) First we show that ${}_R\mathcal{F}$ contains the projective modules if and only if $\rho({}_R\mathcal{F})$ contains the injective modules. Let Φ be the set of all morphisms $\phi \in$ mod-R such that ${}_R\mathcal{F}$ is the intersection of $\operatorname{Ker} T_\phi$; then $\rho({}_R\mathcal{F})$ is the intersection of $\operatorname{Coker} H_\phi$, for all $\phi \in \Phi$. ${}_R\mathcal{F}$ contains the projective (flat) modules if and only if all the morphisms ϕ are monomorphisms and this is equivalent to have that $\rho({}_R\mathcal{F})$ contains the injective modules.

Assume that ${}_R\mathcal{F}$ is closed under kernels of epimorphisms and let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence with $M_1, M \in \rho({}_R\mathcal{F})$. We show that $M_2 \in \rho({}_R\mathcal{F})$. Passing to the duals we have an exact sequence $0 \to M_2^* \to M^* \to M_1^* \to 0$. By Lemma 5.5, M^* and M_1^* belong to ${}_R\mathcal{F}$, hence $M_2^* \in {}_R\mathcal{F}$ by hypothesis and so $M_2 \in \rho({}_R\mathcal{F})$ again by Lemma 5.5. The analogous argument shows that, if $\rho({}_R\mathcal{F})$ is closed under cokernels of monomorphisms, then ${}_R\mathcal{F}$ is closed under kernels of epimorphisms.

The fact that ${}_R\mathcal{F}$ is closed under extensions if and only if $\rho({}_R\mathcal{F})$ is closed under extensions is obtained by dualizing exact sequences and applying Lemma 5.5.

(3) follows by dualizing the exact sequences giving n-submodules and n-images and applying Proposition 5.4 and Lemma 5.5. □

Proposition 5.8. *Let ${}_R\mathcal{F}$ be a definable class of left R-modules and let $\mathcal{T}_R$ be a definable class of right R-modules. If ρ and σ are as in Definition* 5.3*, then the following hold*:

(1) *${}_R\mathcal{F}$ is of cofinite type if and only if $\rho({}_R\mathcal{F})$ is of finite type.*
(2) *$\mathcal{T}_R$ is of finite type if and only if $\sigma(\mathcal{T}_R)$ is of cofinite type.*

Proof. (1) Let Φ be the set of all morphisms $\phi \in$ mod-R such that ${}_R\mathcal{F} = \{N \in R\text{-Mod} \mid \operatorname{Ker} T_\phi(N) = 0\}$.

Assume that ${}_R\mathcal{F}$ is of cofinite type and let $\mathcal{S} = \{S_j \mid j \in J\}$ be a set of compact right R-modules such that $\mathcal{S} \subseteq \mathcal{P}_n$ and ${}_R\mathcal{F} = \mathcal{S}^\intercal$.

For every $S_j \in \mathcal{S}$ let

$$0 \to P_{n,j} \to P_{n-1,j} \to \cdots \to P_{1,j} \to P_{0,j} \to S_j \to 0$$

be a projective resolution of S_j with the modules $P_{i,j}$'s projective finitely generated. For every $1 \leqslant i \leqslant n$, let $\Omega_{i,j}(S_j)$ be the ith syzygy of S_j and let $f_{i,j} : \Omega_{i,j}(S_j) \to P_{i-1,j}$ be the corresponding monomorphism. By dimension shifting and by the fact that all the modules $P_{i,j}$ are projective, we have that $\operatorname{Tor}_i^R(S_j, N) = 0$ if and only if $N \in \operatorname{Ker} T_{f_{i,j}}$, for every $1 \leqslant i \leqslant n$. Hence, ${}_R\mathcal{F}$ is the intersection of the kernels of all the functors $T_{f_{i,j}}$, for all $1 \leqslant i \leqslant n$ and for all $j \in J$. Let $\mathcal{G} = \{f_{i,j} \mid 1 \leqslant i \leqslant n,\ j \in J\}$. The morphisms $f_{i,j}$ are in mod-R thus, the class $\mathcal{T}' = \{M \in \text{Mod-}R \mid \operatorname{Coker} H_{f_{i,j}}(M) = 0,\ 1 \leqslant i \leqslant n,\ j \in J\}$, is a definable class. We have that $\mathcal{G} \subseteq \Phi$, hence $\rho({}_R\mathcal{F})$ is contained in $\mathcal{T}'$. We claim that $\mathcal{T}' = \rho({}_R\mathcal{F})$. In fact, let $M \in \mathcal{T}'$. By Lemma 5.5 M^* belongs to the intersection of the kernels of all the functors $T_{f_{i,j}}$, thus $M^* \in_R \mathcal{F}$. Again by Lemma 5.5 we conclude that $M \in \rho({}_R\mathcal{F})$. Since all the modules $P_{i,j}$ are projective, a dimension shifting argument shows that $\operatorname{Coker} H_{f_{i,j}}(M) = 0$ if and only if $\operatorname{Ext}_R^i(S_j, M) = 0$ for all $1 \leqslant i \leqslant n$ and for all $j \in J$. Thus, $\rho({}_R\mathcal{F}) = \mathcal{S}^\perp$, hence, $\rho({}_R\mathcal{F})$ is of finite type.

Conversely, if $\rho({}_R\mathcal{F})$ is of finite type, there is a set $\mathcal{S} \subseteq \mathcal{P}_n$ of compact right R-modules such that $\rho({}_R\mathcal{F}) = \mathcal{S}^\perp$.

Going backward with the same arguments as above, we conclude that ${}_R\mathcal{F}$ is of cofinite type.

(2) follows by (1) and by Proposition 5.4. □

6. Finite and cofinite type

A definable class is a precovering class, by Theorem 2.5. If it is furthermore closed under extensions and contains $\mathcal{P}_0$, then it is a special precovering class by Theorem 2.2 and by Wakamatsu's Lemma.

This is the reason why a resolving definable class closed under n-submodules is an n-cotilting class (see Theorem 6.1).

On the opposite side, a definable class is a preenveloping class by Theorem 2.4, but in general it is not an enveloping class, that is the dual of Theorem 2.2 does not hold in general. In fact, there exist 1-tilting classes, hence definable and preenveloping, which are not enveloping (see [32, Theorem 3.5]). A definable class closed under extensions and containing $\mathcal{I}_0$ is a preenveloping class, but we cannot apply Wakamatsu's Lemma to conclude that it is a special preenveloping class.

The results in this section, will show that there may exist definable classes closed under extensions and containing $\mathcal{I}_0$ which are not special preenveloping.

More precisely, we will prove that over rings admitting n-cotilting classes which are not of cofinite type, there exist coresolving, definable classes closed under n-images which are not special preenveloping. This will be achieved by showing that these classes are not n-tilting classes (cf. Theorem 3.2). Moreover, if one imposes to such classes the extra condition to be special preenveloping, then one gets that they are of finite type (see Theorem 6.3) and this becomes equivalent to the extra condition on an n-cotilting classes to be of cofinite type (see Corollary 6.4).

We first note a result valid for n-cotilting classes. As it will be shown in the sequel, the corresponding result does not hold for n-tilting classes.

Theorem 6.1. *Let $n \in \mathbb{N}$ and let $\mathcal{F}$ be a class of R-modules. The following are equivalent.*

(1) *$\mathcal{F}$ is an n-cotilting class.*
(2) *$\mathcal{F}$ is resolving, definable and $\mathcal{F}^{\perp} \subseteq \mathcal{I}_n$.*
(3) *$\mathcal{F}$ is resolving, definable and closed under n-submodules.*

In particular a torsion free class is a 1*-cotilting class if and only if it is definable and contains the projective modules.*

Proof. (1) ⇒ (2) As already recalled, the pure injectivity of n-cotilting modules [8,31] imply that n-cotilting classes are definable. Hence, the implication follows by Theorem 3.4.

(2) ⇒ (1) In view of Theorem 3.4, it is enough to show that $\mathcal{F}$ is a covering class. This follows by Proposition 4.3 (3) and Corollary 2.6.

(1) ⇒ (3) We have only to show that $\mathcal{F}$ is closed under n-submodules. By assumption $\mathcal{F} = {}^{\perp}C$, for an n-cotilting module C. Let

$$0 \to A \xrightarrow{f_n} N_{n-1} \xrightarrow{f_{n-1}} \cdots \to N_1 \xrightarrow{f_1} N_0$$

be an exact sequence with $N_i \in \mathcal{F}$, for every $0 \leqslant i \leqslant n-1$. We show that $\mathrm{Ext}^i_R(A, C) = 0$ for every $i \geqslant 1$. By dimension shifting we have $\mathrm{Ext}^i_R(A, C) \cong \mathrm{Ext}^{n+i}_R(\mathrm{Coker}\, f_1, C)$ and for every $i \geqslant 1$, $\mathrm{Ext}^{n+i}_R(-, C) = 0$, since $C \in \mathcal{I}_n$.

(3) ⇒ (2) Since $\mathcal{F}$ contains the projective modules and is closed under n-submodules, $\mathcal{F}$ contains the nth-syzygy $\Omega_n(M)$ of every R-module M. Let $B \in \mathcal{F}^{\perp}$; by dimension shifting we have $\mathrm{Ext}^{n+1}_R(M, B) \cong \mathrm{Ext}^1_R(\Omega_n(M), B) = 0$, for every R-module M, hence $\mathcal{F}^{\perp} \subseteq \mathcal{I}_n$.

The last statement is the equivalence (1) ⇔ (3) for the case $n = 1$. □

As a counterpart of the preceding result for coresolving classes we have:

Proposition 6.2. *Let $n \in \mathbb{N}$ and let $\mathcal{T}$ be a definable class of right R-modules. Let σ be as in Definition 5.3. Consider the following conditions*:

(1) *$\sigma(\mathcal{T})$ is an n-cotilting class.*
(2) *$\mathcal{T}$ is coresolving and closed under n-images.*
(3) *$\mathcal{T}$ is coresolving and ${}^{\perp}\mathcal{T} \subseteq \mathcal{P}_n$.*

Then, (1) ⇔ (2) *and* (2) ⇒ (3).

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows by Proposition 5.7 and by Theorem 6.1.

(2) $\Rightarrow$ (3) Let $A \in {}^{\perp}\mathcal{T}$. For every right R-module M consider its nth-cosyzygy $\Omega_{-n}(M)$. Since $\mathcal{T}$ contains the injective modules, $\Omega_{-n}(M)$ is an n-image of objects in $\mathcal{T}$, hence it belongs to $\mathcal{T}$ by assumption. By dimension shifting we have $\mathrm{Ext}_R^{n+1}(A, M) \cong \mathrm{Ext}_R^1(A, \Omega_{-n}(M)) = 0$, thus $A \in \mathcal{P}_n$. $\square$

Now we are in a position to show that the analogue of Theorem 6.1 formulated for n-tilting classes does not hold. This witnesses the asymmetry, mentioned in the Introduction, between the notion of finite type and cofinite type for n-tilting and n-cotilting classes and between the notion of special precovering and special preenveloping classes.

Theorem 6.3. *Let $n \in \mathbb{N}$ and assume that $\mathcal{T}$ is a definable class of right R-modules. Let σ be the correspondence defined in Definition* 5.3. *The following are equivalent*:

(1) *$\mathcal{T}$ is an n-tilting class.*
(2) *$\sigma(\mathcal{T})$ is an n-cotilting class of cofinite type.*
(3) *$\mathcal{T}$ is coresolving, special preenveloping and closed under n-images.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows by Proposition 3.7 and Proposition 5.8.

(1) $\Rightarrow$ (3) By Theorem 3.2 we have only to show that $\mathcal{T}$ is closed under n-images. Since $\mathcal{T}$ is an n-tilting class we have that $\mathcal{T} = T^{\perp}$ for an n-tilting module T. Consider an exact sequence

$$M_{n-1} \xrightarrow{f_{n-1}} \cdots \to M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} B \to 0$$

with $M_i \in \mathcal{T}$, for every $0 \leqslant i \leqslant n-1$. We show that $\mathrm{Ext}_R^i(T, B) = 0$, for every $i \geqslant 1$. By dimension shifting we have $\mathrm{Ext}_R^i(T, B) \cong \mathrm{Ext}_R^{n+i}(T, \mathrm{Ker}\, f_{n-1})$ and for every $i \geqslant 1$, $\mathrm{Ext}_R^{n+i}(T, -) = 0$, since $T \in \mathcal{P}_n$.

(3) $\Rightarrow$ (1) follows by the implication (2) $\Rightarrow$ (3) in Proposition 6.2 and by Theorem 3.2. $\square$

Summarizing the results proved in this section we can state the following.

Corollary 6.4. *Let $\mathcal{F}$ be an n-cotilting class of left R-modules. Then, $\rho(\mathcal{F})$ is a definable coresolving class closed under n-images.*

Moreover, $\rho(\mathcal{F})$ is an n-tilting class if and only if $\mathcal{F}$ is an n-cotilting class of cofinite type.

Remark 3. Corollary 6.4 shows that even if $\mathcal{F}$ is an n-cotilting class, hence special precovering, $\rho(\mathcal{F})$ will not be a special preenveloping class in general.

7. Definable Torsion classes and definable Torsion free-classes

In this section we specialize the previous results to the case $n = 1$.

Theorem 6.3 stated for $n = 1$ becomes:

Theorem 7.1. *Let $\mathcal{T}$ be a torsion class of right modules which is definable and contains the injective modules. Let σ be the correspondence defined in Definition* 5.3. *Then, the following hold*:

(1) *$\sigma(\mathcal{T})$ is a* 1*-cotilting torsion free class.*
(2) *$\mathcal{T}$ is a* 1*-tilting class if and only if $\sigma(\mathcal{T})$ is a* 1*-cotilting class of cofinite type.*

Remark 4. In [14, Theorem 3.7] it is proved that over artin algebras every definable torsion class containing the injective modules is a 1-tilting class. This result is a particular case of Theorem 7.1, since by [33, Theorem 4.14] (see also [22, Theorem 8.2.8]) every 1-cotilting class over artin algebras is of cofinite type.

We recall that there are 1-cotilting classes which are not of cofinite type. In [9] it is proved that every 1-cotilting class over a valuation domain R is of cofinite type if and only if R is strongly discrete, that is every nonzero prime ideal of R is not idempotent. Thus, by Theorem 7.1 it is possible to exhibit a wide class of examples of definable torsion classes containing the injectives which are not 1-tilting classes.

We want to give explicitly an example of this phenomenon which is, in some sense, a minimal one.

We fix some notations. Let R be a commutative domain and M an R-module. We denote by $t(M)$ the torsion submodule of M and $d(M)$ the divisible submodule of M. For every $r \in R$, we denote by $M[r] = \{x \in M \mid rx = 0\}$. For other unexplained terminology or results on valuation domains, see [16].

Proposition 7.2. *Let R be a maximal valuation domain with idempotent maximal ideal P. Let*

$$\mathcal{F} = \bigl\{N \in R\text{-Mod} \bigm| t(N) \textit{ is an } R/P\text{-module}\bigr\}.$$

$\mathcal{F}$ is a 1-cotilting class not of cofinite type. If ρ is the correspondence defined in Definition 5.3, then $\rho(\mathcal{F})$ is a definable torsion class containing the injective modules which is not a tilting class.

Proof. In [9, Proposition 4.5] it is proved that $\mathcal{F}$ is a 1-cotilting class corresponding to the 1-cotilting module $C = Q \oplus R \oplus R/P$, where Q is the quotient field of R. Thus $\mathcal{F}$ is a definable class, but $\mathcal{F}$ is not of cofinite type. In fact, the class $\mathcal{S}$ of the finitely presented modules in ${}^{\mathsf{T}}\mathcal{F}$ coincides with the class of finitely generated projective modules, thus $\mathcal{S}^{\mathsf{T}}$ is the class of all R-modules.

By Proposition 5.7, $\rho(\mathcal{F})$ is a definable torsion class containing the injective modules and by Corollary 6.4 it is not a tilting class. $\square$

We want now to describe explicitly the torsion class $\rho(\mathcal{F})$ of Proposition 7.2 and we want also to exhibit two sets of coherent functors such that $\mathcal{F}$ and $\rho(\mathcal{F})$ are the intersection of the kernels of these sets of functors.

Proposition 7.3. *Let R and $\mathcal{F}$ be as in Proposition 7.2. Let Φ be the set of all monomorphisms $\{\phi : A \to B \mid \phi \in \text{mod-}R\}$ such that the induced homomorphisms $\bar{\phi} : A/PA \to B/PB$ are monomorphisms. The following hold*:

(1) $N \in \mathcal{F}$ *if and only if* $N \in \operatorname{Ker} T_\phi(N) = 0$, for all $\phi \in \Phi$.
(2) $\rho(\mathcal{F}) = \{M \in \text{Mod-}R \mid \operatorname{Coker} H_\phi(M) = 0$, *for all* $\phi \in \Phi\}$.
(3) $\rho(\mathcal{F}) = \{M \in \text{Mod-}R \mid M/d(M)$ *is an* R/P-module$\}$.

Proof. (1) Let $\mathcal{F}' = \{N \in R\text{-Mod} \mid \operatorname{Ker} T_\phi(N) = 0$, for all $\phi \in \Phi\}$. We show that $\mathcal{F}' = \mathcal{F}$.

First of all we note that a module N belongs to $\mathcal{F}'$ if and only if $t(N) \in \mathcal{F}'$. In fact, for every R-module N the sequence $0 \to t(N) \to N \to N/t(N) \to 0$ is a pure exact sequence, since over valuation domains all torsion free modules are flat. Thus we get the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes t(N) \arrow[r] \arrow[d, "\phi\otimes_R 1_{t(N)}"'] & A \otimes N \arrow[r] \arrow[d, "\phi\otimes_R 1_N"] & A \otimes_R N/t(N) \arrow[r] \arrow[d, "\phi\otimes_R 1_{N/t(N)}"] & 0 \\
0 \arrow[r] & B \otimes t(N) \arrow[r] & B \otimes N \arrow[r] & B \otimes_R N/t(N) \arrow[r] & 0
\end{tikzcd}$$

showing that $\operatorname{Ker} T_\phi(N) = 0$ if and only if $\operatorname{Ker} T_\phi(t(N)) = 0$, for all $\phi \in \Phi$.

By assumption $\operatorname{Ker} T_\phi(N) = 0$, for every R/P-module N and for every $\phi \in \Phi$, hence $\mathcal{F} \subseteq \mathcal{F}'$.

To prove the other inclusion it is enough to show that if N is a torsion module in $\mathcal{F}'$, then N is an R/P-module.

Consider an ordered triple (r,s,t) of nonzero elements of P such that $r=bt$ and $s=ar$, for some $a,b\in P$ and let $c\in P$ be a nonzero element. Let

$$\psi : R/rR\oplus R\to R/sR\oplus R/tR\oplus R$$

be the morphism defined by

$$\psi(x+rR,0)=(ax+sR,x+tR,0);\qquad \psi(0,y)=(y+sR,y+tR,cy).$$

We say that ψ is associated to the quadruple (r,s,t,c). It is easy to check that ψ is a monomorphism, since $s=ar$. Let $\mathcal{G}$ be the set of all morphisms ψ defined as above for all the quadruples (r,s,t,c). Note that $\psi\otimes 1_{R/P}$ is a monomorphism, for all $\psi\in\mathcal{G}$, hence $\mathcal{G}\subseteq\Phi$.

Let $N\in\mathcal{F}'$ be a torsion module; then in particular, $\operatorname{Ker}T_\psi(N)=0$ for all $\psi\in\mathcal{G}$.

Assume by way of contradiction that N is not annihilated by P. Then, there exists a nonzero element $t\in P$ such that $tN\neq 0$. We distinguish two cases:

(a) tN is not a divisible module.
Then there exists $0\neq b\in P$ such that $btN<tN$. Let $x\in tN, x\notin btN$ and let $0\neq a\in\operatorname{Ann}x$. Let $r=bt$ and $s=ar$ and choose an arbitrary $0\neq c\in P$. Consider the morphism ψ associated to the quadruple (r,s,t,c). Then $\psi\otimes 1_N(x+rN,0)=(ax+sN,x+tN,0)$. The element $(ax+sN,x+tN,0)$ is 0 since $ax=0$ and $x\in tN$, but $x+rN$ is nonzero, since $x\notin btN=rN$. So $\operatorname{Ker}T_\psi(N)\neq 0$, a contradiction.

(b) Assume that tN is divisible. Let $0\neq y\in tN$ and let $0\neq c\in P$ be such that $cy=0$. Choose two nonzero elements $a,b\in P$ and let $r=bt$ and $s=ar$. Consider the morphism ψ associated to the quadruple (r,s,t,c). Then, $\psi\otimes 1_N(0,y)=(y+sN,y+tN,0)$. But $tN=sN$, since tN is divisible and $s=abt$, hence $\psi\otimes 1_N(0,y)=0$, since $y\in tN$, a contradiction.

Thus, we conclude that $\mathcal{F}=\mathcal{F}'$.

(2) follows by (1) and by the definition of the correspondence ρ.

(3) Let

$$\mathcal{T}=\bigl\{M\in\text{Mod-}R \bigm| M/d(M)\text{ is an }R/P\text{-module}\bigr\}.$$

We prove that $\mathcal{T}=\rho(\mathcal{F})$.

By (2)

$$\rho(\mathcal{F})=\bigl\{M\in\text{Mod-}R \bigm| \operatorname{Coker}H_\phi(M)=0,\ \text{for all }\phi\in\Phi\bigr\}.$$

First of all note that every divisible module D over a valuation domain is FP-injective, that is $\operatorname{Ext}^1_R(X,D)=0$ for every finitely presented module X. Hence $\operatorname{Coker}H_\phi(D)=0$ for every monomorphism $\phi\in\text{mod-}R$, that is $\rho(\mathcal{F})$ contains all divisible modules. Moreover, for every R-module M the exact sequence $0\to d(M)\to M\to M/d(M)\to 0$ gives rise to the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(A,d(M)) \arrow[r] \arrow[d, "\operatorname{Hom}(\phi,d(M))"'] & \operatorname{Hom}(A,M) \arrow[r] \arrow[d, "\operatorname{Hom}(\phi,M)"'] & \operatorname{Hom}(A,M/d(M)) \arrow[r] \arrow[d, "\operatorname{Hom}(\phi,M/d(M))"] & 0 \\
0 \arrow[r] & \operatorname{Hom}(B,d(M)) \arrow[r] & \operatorname{Hom}(B,M) \arrow[r] & \operatorname{Hom}(B,M/d(M)) \arrow[r] & 0
\end{tikzcd}$$

for all $\phi\in\Phi$.

Thus, $M\in\rho(\mathcal{F})$ if and only if $M/d(M)$ in $\rho(\mathcal{F})$. Moreover, by the definition of the set Φ, we have that $\operatorname{Coker}H_\phi(M)=0$ for every R/P-module M and every $\phi\in\Phi$. Hence, $\mathcal{T}\subseteq\rho(\mathcal{F})$.

We prove now the other inclusion. If $M \in \rho(\mathcal{F})$, then $\operatorname{Coker} H_\psi(M) = 0$, for every ψ in the set $\mathcal{G}$ defined in (1). If $\psi \in \mathcal{G}$ is associated to a quadruple (r, s, t, c), we have that

$$\operatorname{Hom}(\psi, M) : M[s] \oplus M[t] \oplus M \to M[r] \oplus M$$

is given by

$$\operatorname{Hom}(\psi, M)(x, y, z) = (ax + y, x + y + cz),$$

for every $x \in M[s]$, $y \in M[t]$, $z \in M$. The surjectivity of $\operatorname{Hom}(\psi, M)$ implies in particular that $M = M[s] + cM$, hence $sM = scM$. Thus, if $\operatorname{Coker} H_\psi(M) = 0$, for every $\psi \in \mathcal{G}$, then sM is a divisible module for every $0 \neq s \in P$. This implies that PM is a divisible submodule of M, hence contained in $d(M)$. So $M/d(M)$ is an R/P-module, hence $M \in \mathcal{T}$ and thus $\rho(\mathcal{F}) \subseteq \mathcal{T}$. □

8. Questions

Up to now the existence of cotilting classes which are not of cofinite type is known only for the case of nonstrongly discrete valuation domains (cf. [9, Corollary 4.6]).

On the other hand the only case for which it is known that cotilting classes are necessarily of cofinite type is the case of 1-cotilting classes over rings R which are left noetherian and have the property that the modules of flat dimension at most one belong to $\mathcal{P}_1$; for instance artin algebras (cf. [22, Theorem 8.2.8]).

So our question is:

Question 1.

(1) Are there noetherian rings admitting cotilting classes which are not of cofinite type?
(2) In particular, are there artin algebras admitting n-cotilting classes not of cofinite type for some $n > 1$?

To comment on the above questions we recall a generalization of the notions of the little and the big finitistic dimension of a ring R.

Definition 8.1. The *right* (*left*) *little finitistic dimension*, $\operatorname{f.dim} R$, is the supremum of the projective dimension of the compact right (left) R-modules with finite projective dimension.

The *right* (*left*) *big finitistic dimension*, $\operatorname{F.dim} R$, is the supremum of the projective dimension of the right (left) R-modules with finite projective dimension.

Remark 5. The classical definition of the little finitistic dimension of a ring uses finitely generated modules rather than compact modules. So our definition coincides with the classical one in the case of noetherian rings.

It is well known that there are examples of rings for which $\operatorname{f.dim} R$ is strictly smaller than $\operatorname{F.dim} R$. We recall some cases:

(1) Case of non-Dedekind Prüfer domains.
If R is a Prüfer domain, then $\operatorname{f.dim} R = 1$, since every finitely presented module has projective dimension at most one.
If R is a non-Dedekind Prüfer domain, then there are countably generated ideals with projective dimension one, hence R/I has projective dimension 2 and so $\operatorname{F.dim} R > 1$.

(2) Case of noetherian commutative rings.
A combination of a result by Bass [7] and one by Raynaud Gruson [28] shows that, for a commutative noetherian ring, the big finitistic dimension equals the Krull dimension, while by Auslander–Buchsbaum equality [5], the little finitistic dimension of a local noetherian ring equals its depth. Moreover, the Cohen–Macaulay rings are the noetherian commutative rings for which depth $R =$ Krull dim R. Thus, to exhibit examples of commutative noetherian rings for which $\mathrm{f.dim}\, R$ and $\mathrm{F.dim}\, R$ differ it is enough to consider examples of non-Cohen–Macaulay local rings. In [11, Example 9.2(i)] there is an example of a local noetherian commutative domain with $\mathrm{f.dim}\, R = 1$ and $\mathrm{F.dim}\, R = 2$.
(3) Case of artin algebras.
In [30] Smalø constructs a family of examples of finite dimensional algebras R_n, such that $\mathrm{f.dim}\, R_n = 1$ and $\mathrm{F.dim}\, R_n = n$ for every $n \in \mathbb{N}$.

Remark 6. If a ring has the right little finitistic dimension equal to one and it admits n-cotilting classes (of left modules) for some $n > 1$, then these classes cannot be of cofinite type.

In fact, in this case the compact modules contained in $\mathcal{P}_n$ are of projective dimension at most 1, hence every class of cofinite type is a 1-cotilting class.

Note that for a Prüfer domain R, $\mathrm{f.dim}\, R = 1$, but every pure injective module is of injective dimension at most one, hence there are no n-cotilting classes for $n > 1$.

The phenomenon illustrated in the above remark for the case of Prüfer domains does not occur in the case of artin algebras. That is, if an artin algebra R is such that $\mathrm{f.dim}\, R = 1$ and $\mathrm{F.dim}\, R = n > 1$, there exist pure injective modules of injective dimension $n > 1$. In fact, if M is a module of projective dimension n, then its dual is pure injective and of injective dimension n, since duals of injective modules are flat, hence projective.

In the following more particular question p.d.M (i.d.M) denotes the projective (injective) dimension of a module M.

Question 2. Let R be a finite dimensional algebra such that $\mathrm{f.dim}\, R = 1$ and $\mathrm{F.dim}\, R = 2$. Let $M \in$ Mod-R be such that p.d.$M = 2$. Is the class $M^{\intercal}$ a 2-cotilting class?

Note that $M^{\intercal} = {}^{\perp}M^*$ and i.d.$M^* = 2$. Hence $M^{\intercal}$ is closed under direct limit and pure submodules, since M^* (and every cosyzygy of M^*) is pure injective. Thus, by Theorem 6.1 $M^{\intercal}$ is a 2-cotilting class if and only if it is closed under products.

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